

GROWTH OF VALUES OF BINARY QUADRATIC FORMS AND CONWAY RIVERS

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ABSTRACT. We study the growth of the values of binary quadratic form Q on a binary planar tree as it was described by Conway. We show that the corresponding Lyapunov exponents $\Lambda_Q(x)$ as a function of path determined by $x \in \mathbb{R}P^1$ are twice the values of the corresponding exponents for the growth of Markov numbers [5], except for the paths corresponding to the Conway rivers, when $\Lambda_Q(x) = 0$. The relation with Galois results about continued fraction expansions for quadratic irrationals is explained and interpreted geometrically.

1. INTRODUCTION

In his book “The Sensual (Quadratic) Form” Conway [1] described the following “topographic” way to visualise the values of a binary quadratic form

$$Q(x, y) = ax^2 + hxy + by^2, \quad (x, y) \in \mathbb{Z}^2. \quad (1)$$

Following Conway we define the *superbase* of the integer lattice \mathbb{Z}^2 as a triple $(\pm \mathbf{e}_1, \pm \mathbf{e}_2, \pm \mathbf{e}_3)$ such that $(\mathbf{e}_1, \mathbf{e}_2)$ is a basis of the lattice and

$$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \mathbf{0}.$$

We construct a binary planar tree starting from this superbase, such that each base is identified with an edge and each superbase is identified with a vertex. It is easy to see that the edges and vertices which have some vector in common form a path, and we can therefore label each face by the common vector in the path which bounds it (see Fig. 1). Note that all *primitive* lattice vectors, which are not multiples of any other lattice vectors, appear on this tree.

By taking values of the form Q on the vectors of superbase, we get what Conway called the *topograph* of Q containing the values of Q on all primitive lattice vectors. In particular, if $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, $\mathbf{e}_3 = -(1, 1)$ we have the values

$$Q(\mathbf{e}_1) = a, \quad Q(\mathbf{e}_2) = b, \quad Q(\mathbf{e}_3) = c := a + b + h.$$

One can construct the topograph of Q starting from this triple using the following property of any quadratic form, which Conway called the *arithmetic progression rule* and is known also in geometry as the *parallelogram rule*:

$$Q(\mathbf{u} + \mathbf{v}) + Q(\mathbf{u} - \mathbf{v}) = 2(Q(\mathbf{u}) + Q(\mathbf{v})), \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^2. \quad (2)$$

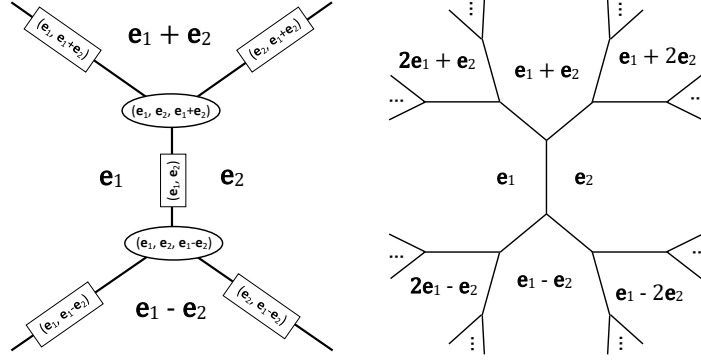


FIGURE 1. The superbase tree and its version with only faces labelled.

Note that quadratic forms (in any dimension) can be characterised as the degree 2 homogeneous functions satisfying the relation (2). This relation leads to the following relation on the topograph of Q (see Fig. 2), which can be used to compute the primitive values of Q recursively.

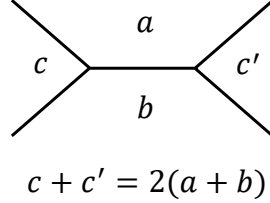


FIGURE 2. Arithmetic progression rule for values of quadratic forms.

In particular, for the standard quadratic form $Q(x, y) = x^2 + y^2$ we have the tree shown on the left of Fig. 3. On the right side of Fig. 3 we show the corresponding *Farey tree*, when at each vertex we have the fractions $\frac{p}{r}, \frac{q}{s}$ and its *Farey mediant* $\frac{p+q}{r+s}$ (see e.g. [2, 5]). Using the Farey tree we can identify the infinite paths γ on a binary tree with real numbers $\xi \in [0, \infty]$ (see more details in the next section). For example, for the golden ratio $\xi = \varphi := \frac{\sqrt{5}+1}{2}$ we have the Fibonacci path shown in bold on both trees of Fig. 3.

We would like to study the growth of the values of Q along the path γ_ξ . More precisely, define

$$\Lambda_Q(\xi) = \limsup_{n \rightarrow \infty} \frac{\ln |Q_n(\xi)|}{n}, \quad (3)$$

where

$$|Q_n(\xi)| = \max(|a_n(\xi)|, |b_n(\xi)|, |c_n(\xi)|)$$

with $(a_n(\xi), b_n(\xi), c_n(\xi))$ being the n -th triple on the path γ_ξ .

For example, for the Fibonacci path with $\xi = \varphi$ we have $q_n = F_{2n}$ being every second Fibonacci number with the growth

$$\limsup_{n \rightarrow \infty} \frac{\ln |Q_n(\xi)|}{n} = \lim_{n \rightarrow \infty} \frac{\ln F_{2n}}{n} = \ln \varphi^2 = 2 \ln \varphi.$$

We will show that a similar result is true for any positive binary quadratic form Q , namely that

$$\Lambda_Q(\xi) = 2\Lambda(\xi), \quad (4)$$

where $\Lambda(\xi)$ is the function introduced in [5] describing the growth of the Markov numbers, or, equivalently, the growth of the monoid $SL_2(\mathbb{N})$.

The monoid $SL_2(\mathbb{N})$ consists of matrices from $SL_2(\mathbb{Z})$ with non-negative entries. They all can be seen on the edges of the Farey tree if we combine two neighbouring fractions $\frac{p}{r}, \frac{q}{s}$ into the matrix

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z}). \quad (5)$$

The function $\Lambda(\xi)$ can be defined as

$$\Lambda(\xi) = \limsup_{n \rightarrow \infty} \frac{\ln \rho(A_n(\xi))}{n}, \quad (6)$$

where $A_n(\xi) \in SL_2(\mathbb{N})$ is attached to the n -th edge along path γ_ξ and $\rho(A)$ is the *spectral radius* of the matrix A , defined as the maximum of the modulus of its eigenvalues [5].

The function $\Lambda(\xi)$ can be extended to $\xi \in \mathbb{RP}^1$ and has very peculiar properties: it is discontinuous everywhere, $GL_2(\mathbb{Z})$ -invariant and takes all real values from $[0, \ln \varphi]$ (see [5]).

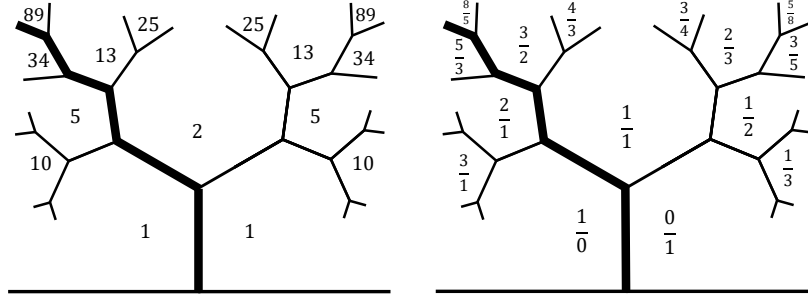


FIGURE 3. Topograph of $Q = x^2 + y^2$ and the corresponding Farey tree with marked “golden” Fibonacci path.

The situation is different for indefinite binary quadratic forms. The reason is the existence of what Conway [1] called the *river*, which, in the case of the forms not representing zero, is an infinite path on the binary tree separating positive and negative values of the form Q (see Fig. 4).

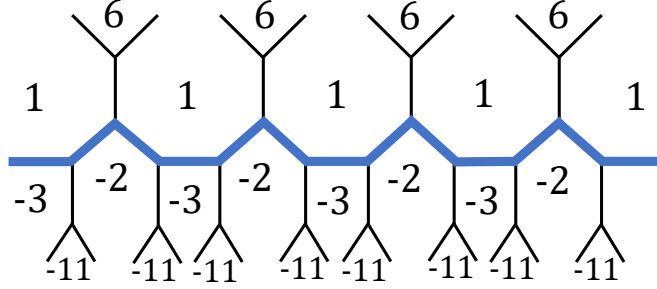


FIGURE 4. Conway river for the indefinite binary quadratic form $Q = x^2 - 2xy - 2y^2$.

Let α_{\pm} be the two real roots of the corresponding quadratic equation

$$Q(\alpha, 1) = 0.$$

Our main result says that for an indefinite form Q not representing zero

$$\Lambda_Q(\xi) = 2\Lambda(\xi), \quad \xi \neq \alpha_{\pm} \quad (7)$$

with $\Lambda_Q(\alpha_{\pm}) = 0 \neq 2\Lambda(\alpha_{\pm})$. We discuss also in more detail the geometry of the corresponding exceptional paths in relation to the continued fraction expansions of the quadratic irrationals α_{\pm} . The Galois result about pure periodic continued fractions naturally appears in this way.

In the case when the indefinite form Q does represent zero (which means that its discriminant is total square) the roots α_{\pm} are rational, so $\Lambda(\alpha_{\pm}) = 0$ and $\Lambda_Q(\xi) = 2\Lambda(\xi)$ for all ξ in this case. Finally in the case of semidefinite forms Q the growth $\Lambda_Q(\xi) \equiv 0$.

2. PATHS IN A BINARY PLANAR TREE, NEGATIVE CONTINUED FRACTIONS AND GALOIS THEOREM

Let us first describe in more detail the correspondence between $\xi \in [0, \infty]$ and paths γ in the planar binary rooted tree \mathfrak{T}_+ , which is shown on the right of Fig. 3. Indeed, the corresponding Farey tree has a unique path γ_{ξ} for any irrational positive ξ such that the limits of the adjacent fractions is ξ (for rational ξ such a path is finite and leads to the corresponding fraction).

In terms of the corresponding continued fraction expansion

$$\xi = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \ddots}}} = [c_0, c_1, c_2, c_3 \dots], \quad c_0 \geq 0, c_i > 0,$$

the path γ_{ξ} starts from the root and can be described as c_0 left-turns on the tree, followed by c_1 right-turns, followed by c_2 left-turns, etc. The

corresponding matrix $A_n(\gamma)$, $\gamma = \gamma_\xi$ is the product of the first n matrices along path γ :

$$A_n(\gamma) = L^{c_0} R^{c_1} L^{c_2} R^{c_3} \dots \quad (8)$$

where

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The matrices L and R freely generate the monoid $SL_2(\mathbb{N})$, which is the positive part of the group $SL_2(\mathbb{Z})$, acting on the rooted tree \mathfrak{T}_+ by left and right turns respectively.

We would like to extend this correspondence to the full binary planar tree \mathfrak{T} , which is known to be the dual tree for the Farey tessellation shown on the left of Fig. 5 (see e.g. Hatcher [2], where we have borrowed this picture from with author's permission).

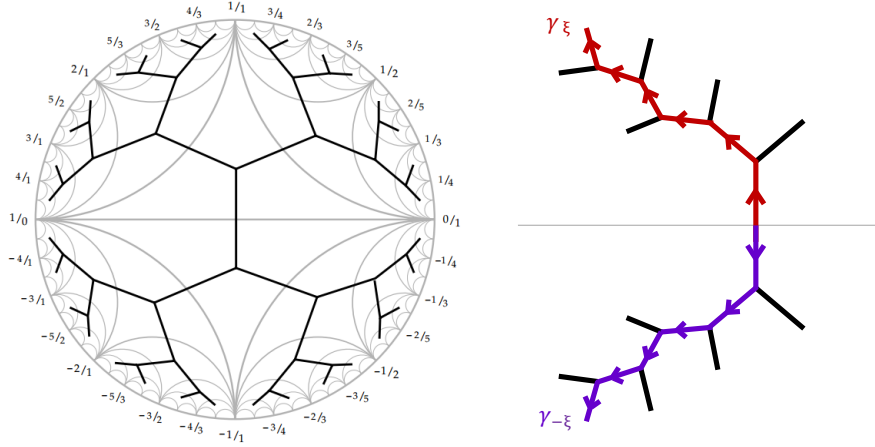


FIGURE 5. Dual tree for Farey tessellation and reflected path $\gamma_{-\xi}$

For this we need to add to L and R the element

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

acting on the full tree \mathfrak{T} as the reflection. Note that since $S^2 = -I$ the corresponding action group is actually the modular group

$$PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\pm I.$$

Since $S = RL^{-1}R$ it can be thought of as a kind of ‘three-point turn’ on the tree [7].

The matrix $A_n(\bar{\gamma})$ for the reflected path $\bar{\gamma} = \gamma_{-\xi}$ can be written then as

$$A_n(\bar{\gamma}) = S A_n(\gamma) S^{-1},$$

or, because of the identities

$$SRS^{-1} = L^{-1}, \quad SLS^{-1} = R^{-1},$$

more explicitly as

$$A_n(\bar{\gamma}) = SL^{c_0} R^{c_1} L^{c_2} R^{c_3} \dots S^{-1} = R^{-c_0} L^{-c_1} R^{-c_2} L^{-c_3} \dots$$

This means that when we move in the negative direction, we need to swap left and right (see right of Fig. 4).

To be consistent with this it is natural to consider for negative real numbers $\eta = -\xi < 0$ with positive $\xi = [c_0, c_1, c_2, c_3 \dots]$, $c_0 \geq 0, c_i > 0$ the *negative continued fraction expansions*

$$\eta = -[c_0, c_1, c_2, c_3 \dots] = [-c_0, -c_1, -c_2, -c_3 \dots]. \quad (9)$$

Note that one can always avoid the negative c_i with $i > 0$ because of the identity

$$-[c_0, c_1, c_2, c_3, \dots] = [-c_0 - 1, 1, c_1 - 1, c_2, c_3, \dots], \quad (10)$$

but we are not going to do this.

This is related to the following Galois result.

The famous Lagrange's theorem [4] says that any quadratic irrational

$$\alpha = \frac{A \pm \sqrt{D}}{B}$$

with $A \in \mathbb{Z}, B, D \in \mathbb{N}$ and D not total square, has a periodic continued fraction expansion

$$\alpha = [a_0, \dots, a_k, \overline{b_1, \dots, b_l}],$$

and conversely, any periodic continued fraction represents a quadratic irrational.

Less known is the following addition due to Galois (see e.g. [4]) characterising *pure periodic continued fraction expansions*

$$\alpha = [\overline{b_1, \dots, b_l}].$$

Galois's theorem. *A quadratic irrational $\alpha = \frac{A+\sqrt{D}}{B}$ has a pure periodic continued fraction expansion*

$$\alpha = [\overline{b_1, \dots, b_l}]$$

if and only if its conjugate

$$\bar{\alpha} = \frac{A - \sqrt{D}}{B}$$

satisfies the inequality

$$-1 < \bar{\alpha} < 0.$$

Moreover, in that case

$$\bar{\alpha} = -[0, \overline{b_l, \dots, b_1}].$$

As we will see later, geometrically the conjugate $\bar{\alpha}$ corresponds to the path going backwards along the infinite periodic Conway river defined by α .

3. CONTINUED FRACTION EXPANSION OF QUADRATIC IRRATIONALS AND THEIR CONJUGATES

To describe our main result we need an answer to the following natural question. Assume that we know the continued fraction expansion of a quadratic irrational

$$\alpha = [a_0, \dots, a_k, \overline{b_1, \dots, b_l}].$$

What is the continued fraction expansion of its conjugate $\bar{\alpha}$?

For example, we have

$$\alpha = \frac{6 + \sqrt{2}}{17} = [0, 2, 3, \overline{2}], \quad \bar{\alpha} = \frac{6 - \sqrt{2}}{17} = [0, 3, 1, \overline{2}]. \quad (11)$$

What is the general rule here?

We could not find the answer to this question in the literature, except in the Galois case, so we present it in this section.

Note first that we can assume that in (20) $a_k \neq b_l$, because otherwise α can be rewritten with the period $[\overline{b_l, b_1, \dots, b_{l-1}}]$.

Proposition 1. *Let $\alpha = [a_0, a_1, \dots, a_k, \overline{b_1, \dots, b_l}]$ be the continued fraction expansion of a quadratic irrational with $a_k < b_l$, and $k \geq 1$. Then the continued fraction expansion of its conjugate is*

$$\bar{\alpha} = [a_0, \dots, a_{k-1} - 1, 1, b_l - a_k - 1, \overline{b_{l-1}, b_{l-2}, \dots, b_1, b_l}]. \quad (12)$$

Let $a_k > b_l$, $k \geq 1$, then

$$\bar{\alpha} = [a_0, \dots, a_{k-1}, a_k - b_l - 1, 1, b_{l-1} - 1, \overline{b_{l-2}, b_{l-3}, \dots, b_1, b_l, b_{l-1}}]. \quad (13)$$

Here it will be convenient for us to allow the zeros in the continued fraction expansion, which can be always avoided using the identities

$$[c_0, \dots, c_i, 0, c_{i+1}, c_{i+2}, c_{i+3} \dots] = [c_0, \dots, c_i + c_{i+1}, c_{i+2}, c_{i+3} \dots], \quad (14)$$

$$[c_0, \dots, c_i, 0, 0, c_{i+1}, c_{i+2}, c_{i+3} \dots] = [c_0, \dots, c_i, c_{i+1}, c_{i+2}, c_{i+3} \dots]. \quad (15)$$

In particular, in the example (11) the formula (13) gives for $\alpha = [0, 2, 3, \overline{2}]$

$$\bar{\alpha} = [0, 2, 0, 1, 1, \overline{2}] = [0, 3, 1, \overline{2}].$$

A more sophisticated example:

$$\begin{aligned} \alpha &= \frac{11523 + \sqrt{15006}}{9222} = [1, 3, 1, 4, \overline{7, 2, 3, 9}], \\ \bar{\alpha} &= \frac{11523 - \sqrt{15006}}{9222} = [1, 3, 0, 1, 4, \overline{3, 2, 7, 9}] = [1, 4, 4, \overline{3, 2, 7, 9}]. \end{aligned}$$

Proof. It is enough to consider the case $k = 1$. Let us first define

$$\beta = [\overline{b_1, \dots, b_{l-1}, b_l}], \quad \tilde{\beta} = [\overline{b_{l-1}, b_{l-2}, \dots, b_1, b_l}].$$

Then, by definition,

$$\alpha = [a_0, a_1, \overline{b_1, \dots, b_l}] = a_0 + \frac{1}{a_1 + \frac{1}{\beta}},$$

so that

$$\bar{\alpha} = a_0 + \frac{1}{a_1 + \frac{1}{\bar{\beta}}}.$$

Using the Galois result and identity (10) we have

$$\begin{aligned}\bar{\beta} &= -[0, \bar{b}_l, \dots, b_1] \\ &= [-1, 1, b_l - 1, \overline{b_{l-1}, \dots, b_1, b_l}].\end{aligned}$$

We can now directly compute

$$\begin{aligned}\bar{\alpha} &= a_0 + \frac{1}{a_1 + \frac{1}{\bar{\beta}}} = a_0 + \frac{1}{a_1 + \frac{1}{-1 + \frac{1}{1 + \frac{1}{b_l - 1 + \frac{1}{\bar{\beta}}}}}} \\ &= a_0 + \frac{1}{a_1 + \frac{1}{-1 + \frac{1}{1 + \frac{\tilde{\beta}}{(b_l - 1)\tilde{\beta} + 1}}}} = a_0 + \frac{1}{a_1 + \frac{1}{-1 + \frac{(b_l - 1)\tilde{\beta} + 1}{b_l\tilde{\beta} + 1}}} \\ &= a_0 + \frac{1}{a_1 + \frac{b_l\tilde{\beta} + 1}{-\tilde{\beta}}} = a_0 - \frac{\tilde{\beta}}{(b_l - a_1)\tilde{\beta} + 1} = a_0 - 1 + \frac{\tilde{\beta}(b_l - a_1 - 1) + 1}{(b_l - a_1)\tilde{\beta} + 1} \\ &= a_0 - 1 + \frac{1}{\frac{(b_l - a_1)\tilde{\beta} + 1}{\tilde{\beta}(b_l - a_1 - 1) + 1}} = a_0 - 1 + \frac{1}{1 + \frac{\tilde{\beta}}{\tilde{\beta}(b_l - a_1 - 1) + 1}} \\ &= a_0 - 1 + \frac{1}{1 + \frac{1}{(b_l - a_1 - 1) + \frac{1}{\tilde{\beta}}}} = [a_0 - 1, 1, b_l - a_1 - 1, \overline{b_{l-1}, \dots, b_0, b_l}].\end{aligned}$$

This completes the case $a_k < b_l$. In the case $a_k > b_l$ we have by (14),

$$\alpha = [a_0, \dots, a_k - b_l, 0, \overline{b_l, b_1, \dots, b_{l-1}}],$$

and the result then immediately follows from the previous case with a_k replaced by 0 and a_{k-1} replaced by $a_k - b_l$. \square

Note that formulae (12),(13) determine involutions since

$$\begin{aligned}\bar{\bar{\alpha}} &= [a_0, \dots, a_{k-1} - 1, 1 - 1, 1, b_l - (b_l - a_k - 1) - 1, \overline{b_1, \dots, b_{l-1}}, b_l] \\ &= [a_0, \dots, a_{k-1} - 1, 0, 1, a_k, \overline{b_1, \dots, b_{l-1}}, b_l] \\ &= [a_0, \dots, a_{k-1}, a_k, \overline{b_1, \dots, b_{l-1}}, b_l] = \alpha,\end{aligned}$$

$$\begin{aligned}\bar{\bar{\alpha}} &= [a_0, \dots, a_{k-1}, a_k - b_l - 1, 1 - 1, 1, b_{l-1} - 1 - (b_{l-1} - 1), \overline{b_l, b_1, \dots, b_{l-1}}] \\ &= [a_0, \dots, a_{k-1}, a_k - b_l - 1, 0, 1, 0, \overline{b_l, b_1, \dots, b_{l-1}}] \\ &= [a_0, \dots, a_{k-1}, a_k - b_l, 0, \overline{b_l, b_1, \dots, b_{l-1}}] = \alpha.\end{aligned}$$

Let us consider now the special case $k = 0$.

Proposition 2. *Let $\alpha = [a_0, \overline{b_1, \dots, b_l}]$ with $a_0 < b_l$. Then the conjugate $\bar{\alpha}$ can be given as the negative continued fraction expansion*

$$\bar{\alpha} = -[b_l - a_0, \overline{b_l, \dots, b_1}]. \quad (16)$$

When $a_0 > b_l$ we have

$$\bar{\alpha} = [a_0 - b_l - 1, 1, b_l - 1, \overline{b_{l-1}, b_{l-2}, \dots, b_1, b_l}]. \quad (17)$$

Proof. We have

$$[a_0, \overline{b_1, \dots, b_n}] = a_0 + [0, \overline{b_0, \dots, b_n}],$$

so by the Galois result

$$\begin{aligned}\overline{a_0 + [0, \overline{b_1, \dots, b_l}]} &= a_0 - [\overline{b_l, \dots, b_1}] \\ &= -[b_l - a_0, \overline{b_{l-1}, \dots, b_1, b_l}].\end{aligned}$$

In the case when $a_0 > b_l$, we can rewrite

$$[a_0, \overline{b_1, \dots, b_l}] = [a_0 - b_l, 0, \overline{b_l, b_1, \dots, b_{l-1}}]$$

and proceed as in the case when $k \geq 1$ to get the formula (17). \square

Note that only in the Galois pure periodic case and in the case with $k = 0, a_0 < b_l$ do we have to use negative continued fractions. We will see now that these are the only cases when the initial position is on the Conway river.

4. PATHS TO CONWAY RIVER

Let

$$Q(x, y) = ax^2 + hxy + by^2$$

be an indefinite binary quadratic form not representing zero, meaning that $Q(x, y) \neq 0$ for all $(x, y) \in \mathbb{Z}^2 \setminus (0, 0)$. Equivalently, this means that the discriminant of the form

$$D = h^2 - 4ab \quad (18)$$

is positive, but not a total square.

In that case Conway [1] showed that on the topograph of Q positive and negative values are separated by a unique periodic river (see Fig. 4 above).

We are now going to explain how the continued fractions determine the path from Q (identified with the corresponding vertex on the topograph) to the Conway river and the relation with the Galois result.

Let us assume for simplicity that all the values

$$a = Q(1, 0), \quad b = Q(0, 1), \quad c = Q(1, 1) = a + b + h$$

are of the same sign (say, positive), otherwise Q is already on the river. Let us assume also that $h < 0$, so we are going down to the river.

Let $\alpha, \bar{\alpha}$ be the real roots of the quadratic equation

$$Q(\alpha, 1) = a\alpha^2 + h\alpha + b = 0. \quad (19)$$

Again for simplicity let us assume that

$$\alpha = \frac{-h + \sqrt{D}}{2a}, \quad D = h^2 - 4ab$$

is the dominant (maximal modulus) root, so that the conjugate is

$$\bar{\alpha} = \frac{-h - \sqrt{D}}{2a}.$$

Let

$$\alpha = [a_0, a_1, \dots, a_k, \overline{b_1, \dots, b_l}] \quad (20)$$

be the continued fraction expansion of α .

From the general theory of the Farey tree and continued fractions we have the following result.

Proposition 3. *The continued fraction expansion (20) describes the unique path γ_α going from Q to the corresponding Conway river, and then along the river to the boundary of the tree determined by α . A similar path $\gamma_{\bar{\alpha}}$ leading to $\bar{\alpha}$ is described by the Propositions 1 and 2 above.*

The Conway river of Q is an infinite in both directions path from $\bar{\alpha}$ to α described by the periodic part $[\overline{b_1, \dots, b_l}]$ of the expansion.

The finite path π from Q to the Conway river can be given for $k \geq 1$ by

$$\pi = \begin{cases} [a_0, \dots, a_k - b_l - 1] & \text{if } a_k > b_l \\ [a_0, \dots, a_{k-1} - 1] & \text{if } a_k < b_l \end{cases}$$

and for $k = 0$ by

$$\pi = \begin{cases} a_0 - b_l & \text{if } a_0 > b_l \\ \emptyset & \text{if } a_0 < b_l \end{cases}.$$

This also gives a geometric interpretation of our formulas from the previous section, which is illustrated in Fig. 7 in the example of $Q = 17x^2 - 12xy + 2y^2$ with

$$\alpha = \frac{6 + \sqrt{2}}{17} = [0, 2, 3, \bar{2}], \quad \bar{\alpha} = \frac{6 - \sqrt{2}}{17} = [0, 3, 1, \bar{2}].$$

Let us describe now the special (Galois) case when α is a pure periodic continued fraction.

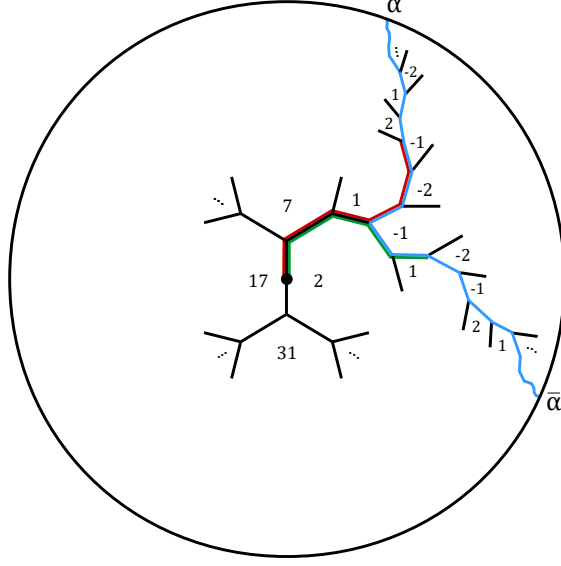


FIGURE 6. Paths to α and $\bar{\alpha}$ and Conway river for $Q = 17x^2 - 12xy + 2y^2$

Let us call the quadratic form $Q(x, y) = ax^2 + hxy + by^2$ *Galois form* if the continued fraction expansion of the dominant root of $Q(\alpha, 1) = 0$ is pure periodic. By the Galois theorem this is equivalent to the conditions $\alpha > 1$, $-1 < \bar{\alpha} < 0$.

Proposition 4. *A binary quadratic form Q is Galois if and only if $Q(1, 0) = a$, $Q(0, 1) = b$, $Q(1, 1) = c$ satisfy*

$$ab < 0, \quad ac < 0, \quad ac' = a(2a + 2b - c) > 0.$$

The proof is obvious geometrically: $\alpha > 1$ and $-1 < \bar{\alpha} < 0$ is true if and only if the Conway river travels along the edges separating a and c' from b and c , see Fig. 9.

An example of the Galois form is the “golden” form $Q = x^2 - xy - y^2$ corresponding to $a = 1 = c'$, $b = c = -1$ with

$$\alpha = \varphi = \frac{1 + \sqrt{5}}{2} = [\bar{1}], \quad \bar{\alpha} = \frac{1 - \sqrt{5}}{2} = -[0, \bar{1}].$$

5. LYAPUNOV EXPONENTS FOR VALUES OF BINARY FORMS

Now we are ready to state and prove our main results.

Let $Q(x, y) = ax^2 + hxy + by^2$ be a binary quadratic form (definite or indefinite).

If the form Q is indefinite, then we assume for the beginning that Q does not represent zero in the sense that $Q(x, y) \neq 0$ for all $(x, y) \in \mathbb{Z}^2 \setminus (0, 0)$. In

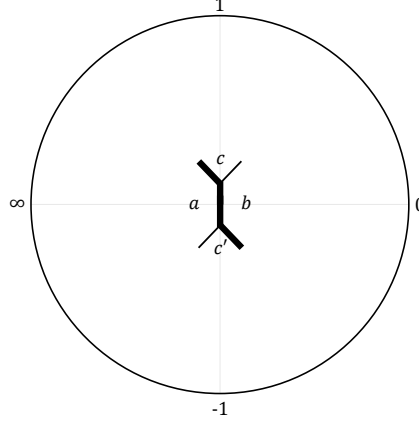


FIGURE 7. Galois form Q on the Conway river

that case the two roots α_{\pm} of the quadratic equation

$$Q(\alpha, 1) = 0$$

are real quadratic irrationals, corresponding to the ends of the Conway river.

To study the growth of the values of Q along the path γ_{ξ} we define the corresponding Lyapunov exponent as

$$\Lambda_Q(\xi) = \limsup_{n \rightarrow \infty} \frac{\ln |Q_n(\xi)|}{n}, \quad |Q_n(\xi)| = \max(|a_n(\xi)|, |b_n(\xi)|, |c_n(\xi)|), \quad (21)$$

where $a_n(\xi), b_n(\xi), c_n(\xi)$ are the values of Q at the n -th superbase on the path γ_{ξ} .

Let $\Lambda(\xi)$ be the Lyapunov exponent of the monoid $SL_2(\mathbb{N})$ introduced in [5]:

$$\Lambda(\xi) = \limsup_{n \rightarrow \infty} \frac{\ln \rho(A_n(\xi))}{n}, \quad (22)$$

where $A_n(\xi) = A_n(\gamma_{\xi})$ are the matrices (8) and $\rho(A)$ is the spectral radius of the matrix A .

Theorem 1. *For the definite binary quadratic forms Q the Lyapunov exponent*

$$\Lambda_Q(\xi) = 2\Lambda(\xi).$$

For the indefinite binary quadratic forms Q not representing 0 we have

$$\Lambda_Q(\xi) = \begin{cases} 2\Lambda(\xi) & \text{if } \xi \neq \alpha_{\pm} \\ 0 & \text{if } \xi = \alpha_{\pm} \end{cases}$$

In other words, the only two exceptional paths with zero growth are those leading to two ends of the Conway river of Q .

Proof. Let us introduce first for the form $Q(x, y) = ax^2 + hxy + by^2$ its matrix defined by

$$Q(x, y) = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & \frac{h}{2} \\ \frac{h}{2} & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

which, slightly abusing the notation, we also denote by Q :

$$Q = \begin{pmatrix} a & \frac{h}{2} \\ \frac{h}{2} & b \end{pmatrix}.$$

Its determinant

$$\det Q = ab - \frac{h^2}{4} = -\frac{D}{4}$$

is minus a quarter of the discriminant $D = h^2 - 4ab$ of the quadratic equation $Q(\xi, 1) = 0$.

The action of the group $SL_2(\mathbb{Z})$ on Q is defined in the matrix form by

$$Q \rightarrow Q' = A^t Q A,$$

where

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z}). \quad (23)$$

This determines a 3-dimensional representation $A \rightarrow \hat{A}$ of $SL_2(\mathbb{Z})$, when A acts on the coefficients (a, h, b) of the form Q by

$$\hat{A} = \begin{pmatrix} p^2 & 2pr & r^2 \\ pq & pr + qs & rs \\ q^2 & 2qs & s^2 \end{pmatrix}.$$

If the eigenvalues of the matrix A are λ and λ^{-1} , then the eigenvalues of \hat{A} are λ^2 , λ^{-2} and 1.

Consider first the case when the coefficients of the form a, b, h are all positive. In that case Conway's Climbing Lemma [1] guarantees the permanent growth whichever upward path we choose (see Fig. 9).

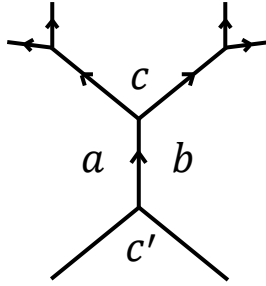


FIGURE 8. Climbing Lemma

We claim that the corresponding growth is given by $\Lambda_Q(\xi) = 2\Lambda(\xi)$.

To prove this introduce two norms on binary forms $Q = ax^2 + hxy + by^2$. The first norm is the one used in the definition of Λ_Q :

$$|Q| = \max(|a|, |b|, |c|), \quad c = a + b + h. \quad (24)$$

The second norm is

$$|Q|_h = \max(|a|, |b|, |h|). \quad (25)$$

Since any two norms in a finite-dimensional space are equivalent, we have for all Q

$$c_1|Q|_h \leq |Q| \leq c_2|Q|_h$$

for some positive constants c_1, c_2 . This means that we can replace $|Q|$ by $|Q|_h$ in the definition

$$\Lambda_Q(\xi) = \limsup_{n \rightarrow \infty} \frac{\ln |Q_n(\xi)|}{n} = \limsup_{n \rightarrow \infty} \frac{\ln |Q_n(\xi)|_h}{n}.$$

Now let γ_ξ be a path and

$$A_n(\xi) = \begin{pmatrix} p_n & q_n \\ r_n & s_n \end{pmatrix} \in SL_2(\mathbb{Z}).$$

be the corresponding matrices (8), so that the matrices of the forms along the path are

$$Q_n(\xi) = A_n(\xi)^T Q A_n(\xi),$$

or, explicitly

$$\begin{pmatrix} a_n \\ h_n \\ b_n \end{pmatrix} = \begin{pmatrix} p_n^2 & 2p_n r_n & r_n^2 \\ p_n q_n & p_n r_n + q_n s_n & r_n s_n \\ q_n^2 & 2q_n s_n & s_n^2 \end{pmatrix} \begin{pmatrix} a \\ h \\ b \end{pmatrix}. \quad (26)$$

Recall also that the function $\Lambda(\xi)$ can be alternatively defined as

$$\Lambda(\xi) = \limsup_{n \rightarrow \infty} \frac{\ln w_n(x)}{n},$$

where $w_n = r_n + s_n$ (see Proposition 1 in [5]).

Similarly to [5] assume without loss of generality that $\xi \in [0, 1]$ and consider the cases $\xi = 0$ and $\xi > 0$ separately. When $\xi = 0$ we have

$$A_n(0) = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},$$

which determines the quadratic growth of $|Q_n|_h$ in n and as a result

$$\Lambda_Q(0) = 0 = 2\Lambda(0).$$

If $0 < \xi \leq 1$ we have $p_n \leq r_n, q_n \leq s_n$ and thus from (26)

$$\begin{aligned} |Q_n(\xi)|_h &\leq \max((p_n + r_n)^2, (p_n + s_n)(q_n + r_n), (q_n + s_n)^2) |Q|_h \\ &\leq \max(4r_n^2, (r_n + s_n)^2, 4s_n^2) |Q|_h \leq 4(r_n + s_n)^2 |Q|_h = 4w_n^2 |Q|_h. \end{aligned}$$

On the other hand by assumption the initial a, b, h are all positive, so, since they are also integer, $a, b, h \geq 1$. From (26) then it follows that

$$|Q_n(\xi)|_h \geq (p_n + r_n)^2 + (p_n + s_n)(q_n + r_n) + (q_n + s_n)^2 > r_n^2 + s_n^2 \geq \frac{1}{2}(r_n + s_n)^2 = \frac{1}{2}w_n^2.$$

Thus we have

$$\frac{1}{2}w_n^2 \leq |Q_n(\xi)|_h \leq 4w_n^2|Q|_h,$$

which implies that

$$\Lambda_Q(\xi) = \limsup_{n \rightarrow \infty} \frac{\ln |Q_n(x)|}{n} = \limsup_{n \rightarrow \infty} \frac{\ln w_n^2(x)}{n} = 2\Lambda(\xi).$$

This completes the proof in the growing case. Assume now that the path goes down, so the values a, b, c may decrease.

Consider first the case when the form Q is positive definite. Then all the values are positive, so it is clear that for any path at some point the growth will start again and we can repeat our arguments to get the claim.

However, if Q is indefinite this is no longer true, since Q can take negative values as well. By Conway's result [1] positive and negative values of Q are separated by the infinite periodic river.

There are three possibilities: either the path does not cross the river, it crosses the river (or starts on the river and then leaves it), or it is after some point stuck on the river forever.

If the path does not cross the river the values of Q will remain positive and thus bounded from below, so at some point we will have growth and we repeat the arguments to prove the claim in this case as well.

If the path crosses the river then at some point all the values of a_n, b_n, c_n will become negative, and we can repeat the arguments for $-Q$ to get the claim. If we start on the river, then after the point of departure we can use the same arguments as before.

Finally, there are exactly two paths which are stuck on the river, corresponding to $\xi = \alpha_{\pm}$. In that case we have no growth and

$$\Lambda_Q(\alpha_{\pm}) = 0.$$

Note that the corresponding $\Lambda(\alpha_{\pm}) \neq 0$, so $\Lambda_Q(\xi) \neq 2\Lambda(\xi)$ in that case. \square

Let us discuss now the remaining case: indefinite forms representing zero and semidefinite forms with zero discriminant.

Proposition 5. *For indefinite forms Q representing zero*

$$\Lambda_Q(\xi) = 2\Lambda(\xi), \quad \xi \in \mathbb{RP}^1,$$

while for the semidefinite forms

$$\Lambda_Q(\xi) \equiv 0.$$

Proof. The topograph of the indefinite forms representing zero (with discriminant D being total square) is described in Conway [1]. In that case we have two "lakes" corresponding to zero values connected by a finite river

separating positive and negative values of Q . In the exceptional cases the river disappears and two lakes are adjacent, see the First Lecture in [1].

This picture agrees with the fact that the roots α_{\pm} of the corresponding equation

$$Q(\alpha, 1) = 0$$

are rational. A simple analysis shows that the previous arguments in this case work as well, with the only difference that now both $\Lambda_Q(\alpha_{\pm})$ and $\Lambda(\alpha_{\pm})$ are zero, and thus equality $\Lambda_Q(\xi) = 2\Lambda(\xi)$ holds in this case as well.

In the semidefinite case equivalent to the case $Q = ax^2$ we have zero growth, which simply follows from the Conway description [1]. \square

6. CONCLUDING REMARKS

Let us compare this with the growth of the Markov triples considered in [5]. Markov triples are the positive solutions of the Markov equation

$$x^2 + y^2 + z^2 - 3xyz = 0. \quad (27)$$

They all can be found from the obvious one $(1, 1, 1)$ by compositions of Vieta involutions

$$(x, y, z) \rightarrow (x, y, 3xy - z) \quad (28)$$

and permutations of x, y, z .

The right-hand side of the Markov equation

$$F(x, y, z) := x^2 + y^2 + z^2 - 3xyz$$

is cubic polynomial, which is *quadratic* in every variable, implying the Vieta involution. It is also symmetric under the permutations of x, y, z . One can consider also a version of the Markov equation

$$F(x, y, z) = D,$$

which has similar properties (see [5]).

We claim that this paper can be interpreted in a similar way for degree 2 polynomial

$$B(x, y, z) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc.$$

Indeed, the transformation

$$(a, b, c) \rightarrow (a, b, c' = 2a + 2b - c)$$

is nothing but the Vieta involution for the equation

$$a^2 + b^2 + c^2 - 2ab - 2ac - 2bc = D. \quad (29)$$

In fact, $B(a, b, c)$ is nothing other than the discriminant of the binary quadratic form $Q = ax^2 + hxy + by^2$:

$$D = h^2 - 4ab = (c - a - b)^2 - 4ab = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc.$$

Our results say that the Lyapunov exponents for the equation (29) with $D < 0$ are twice the Lyapunov exponents for the Markov dynamics. When $D > 0$ then the same is true with the exception of Conway river paths, for which the Lyapunov exponents vanish.

Note that the equation (29) with $D < 0$ determines the two-sheeted hyperboloid, while for $D > 0$ this is a one-sheeted hyperboloid. The semidefinite (degenerate) forms with $D = 0$ correspond to the cone.

We would like to mention Klein's correspondence [3] between indefinite binary quadratic forms and geodesics in hyperbolic plane, which we have learnt recently from [6]. As we have just seen the projectivized vector space of real binary quadratic forms is a real projective plane with the degenerate forms forming a conic section determined by (29) with $D = 0$. Definite forms correspond to the points inside this conic, which in the Klein model represents the points of the hyperbolic plane. The indefinite forms correspond to the points outside the conic, which by polarity are represented by the hyperbolic lines. On the binary tree these lines are nothing other than Conway rivers.

7. ACKNOWLEDGEMENTS

We are very grateful to Alexey Bolsinov for numerous helpful discussions and to Boris Springborn for sending us a preliminary version of his very interesting work [6].

The work of K.S. was supported by the EPSRC as part of PhD study at Loughborough.

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